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Simplified Solutions for Ablation in a Finite Slab

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Nomenclature

B_1 to B_4 = constants of integration
c = specific heat, Btu/lb-°F
h = combined heat-transfer coefficient, Btu/sec-°F-in.^{2} $= h_c + h_r$
h_c = convective heat-transfer coefficient, Btu/sec-°F-in.^{2}
h_r = radiative heat-transfer coefficient, Btu/sec-°F-in.^{2}
ΔH_{eff} = effective heat of ablation $= L + c(T_0 - T_m)\rho_c/\rho_s$, Btu/lb
k = thermal conductivity, Btu/sec-in.-°F
L = heat of charring, Btu/lb
s = distance of the ablative material being removed away which is function of time, in.
T = temperature, °F
v = ablation velocity, in./sec
x = distance coordinate in direction of ablation, in.
α = thermal diffusivity $= k/c\rho$, in.^{2}/sec
ϵ = porosity of the ablative material
ρ = density, lb/in.^{3}
δ = depth, in.
ξ = distance coordinate $= x - s(\theta)$ (see Fig. 1)
θ = time, sec
λ = proportional constant defined by Eq. (19)
erf = error function

Subscripts

c = char-gas layer
s = virgin material
i = initial
0 = surface
g = recovery gas
m = ablation

In a rocket motor, when heat flows from the exhaust gases to the ablating material, there are two heating periods. The first period raises the surface temperature of the material to the ablation temperature The time required to reach this temperature can be estimated by the method described in Ref. 1. After the surface reaches this temperature, the second period is initiated during which the ablation penetrates into the material at this temperature. Some of the material is continuously removed by being blown away, some becomes charred, and some is unchanged. The model is shown in Fig. 1. In order to simplify the problem, the following assumptions are made: 1) flow is one-dimensional and linear, 2) thermal equilibrium is established in the char-gas layer, 3) heat of reaction of the ablation material is negligible, 4) porosity of the ablative material is negligible, 5) the material is opaque, 6) temperature gradient in the virgin material is negligible [to be used in Eq. (10)], 7) transpiration and diffusion effects of the gases are negligible, and 8) thermal conductivity and thermal diffusivity of the material and the combined heat-transfer coefficient are constant.

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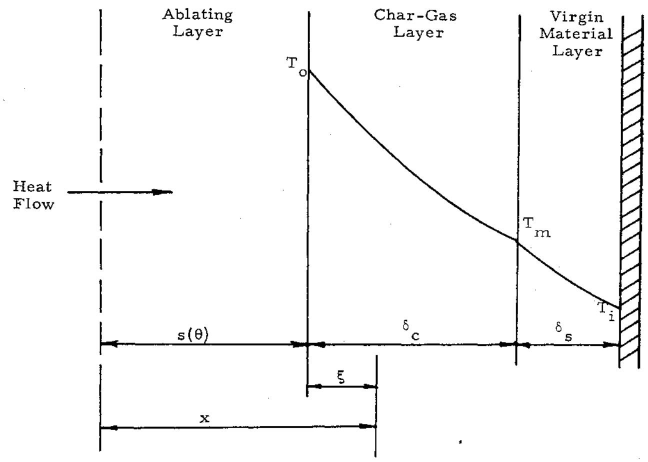

Fig. 1 Ablation model.

The one-dimensional heat conduction equation is

$$\partial T/\partial\theta = \alpha\partial^2T/\partial x^2 \quad (1)$$

Figure 1 shows that

$$\xi = x - s(\theta) \quad (2)$$

where s is a function of time θ. Equation (1) can be transformed to this variable as

$$\partial T/\partial\theta = \alpha(\partial^2T/\partial\xi^2) + ds/d\theta\, \partial T/\partial\xi \quad (3)$$

which is the equation for the gas-char layer. The virgin material layer is governed by Eq. (1). The initial and boundary conditions are

$$T_c(x, 0) = T_s(x, 0) = T_i \quad (4)$$

$$T_c(0, \theta) = T_0 \quad (5)$$

$$T_c(\delta_c, \theta) = T_m \quad (6)$$

$$T_s(\delta_c, \theta) = T_m \quad (7)$$

$$T_s(\delta_s, \theta) = T_i \quad (8)$$

In addition, the heat balances give two more boundary conditions:

$$\text{at } \xi = 0 \qquad -k_c(\partial T/\partial\xi) = h(T_g - T_0) \quad (9)$$

$$\text{at } \xi = \delta_c \qquad k_s(\partial T_s/\partial\xi) - k_c(\partial T_c/\partial\xi) = L\rho_s(ds/d\theta) \quad (10)$$

Because of assumption (6), Eq. (10) becomes

$$\text{at } \xi = \delta_c \qquad -k_c(\partial T_c/\partial\xi) = L\rho_s(ds/d\theta) \quad (11)$$

where L may be defined as the heat of charring, which is analogous to the heat of melting.

Steady-State Solution

Char-gas layer

Equation (3) becomes

$$\alpha\partial^2T/\partial\xi^2 + v\partial T/\partial\xi = 0 \quad (12)$$

where v is the ablation velocity that does not vary with time. The initial and boundary conditions are Eqs. (4–6 and 9) and

$$-k_c(\partial T_c/\partial\xi)_{\delta c} = L\rho_s v \quad (13)$$

which is a form of Eq. (11) under steady-state conditions.

The solution of Eq. (12) with initial and boundary conditions defined by Eqs. (4–6) is

$$\frac{T - T_0}{T_m - T_0} = \frac{1 - \exp\left(\frac{-v\xi}{\alpha_c}\right)}{1 - \exp\left(\frac{-v\delta_c}{\alpha_c}\right)} \tag{14}$$

which is the temperature distribution in the gas-char layer.

Differentiating Eq. (14) with respect to ξ, substituting into Eqs. (9) and (13), and simplifying give

$$v = \frac{h(T_g - T_0)}{\rho_s \Delta H_{\text{eff}}} \tag{15}$$

$$\delta_c = \frac{\alpha_c}{v} \ln \frac{\Delta H_{\text{eff}}}{L} \tag{16}$$

where

$$\Delta H_{\text{eff}} = L + c(T_0 - T_m)\rho_c/\rho_s \tag{17}$$

This ΔH_{eff} is called effective heat of ablation, which differs from L, the heat of charring. It is not the value of heat of reaction, which has been neglected in this derivation.

Virgin material layer

The solution of Eq. (1) under steady state with the boundary conditions (7) and (8) can be easily derived as

$$\frac{T - T_i}{T_m - T_i} = \frac{x - \delta_s}{\delta_c - \delta_s} \tag{18}$$

Transient Solution

Char-gas layer

The governing equations are Eqs. (3–6, 9, and 11). The method of solution is similar to that of the melting problem described by Carslaw and Jaeger[1] which has been applied to ablation by Grosh.[2] The method is as follows. Let

$$s = 2\lambda(\alpha\theta)^{1/2} \tag{19}$$

which means that the moving boundary s is proportional to the square root of the product of the thermal diffusivity and time. λ is a proportional constant that will be determined from the environmental condition and will be described later. Then the ablation velocity is the derivative of Eq. (19) with respect to time or

$$ds/d\theta = \lambda(\alpha/\theta)^{1/2} \tag{20}$$

It has been verified that one of the solutions of Eq. (3) is

$$T = B_1 + B_2 \operatorname{erf}\left(\frac{\xi}{2(\alpha\theta)^{1/2}} + \lambda\right) \tag{21}$$

The constants B_1 and B_2 are determined from boundary conditions (5) and (6). The solution is

$$\frac{T - T_0}{T_m - T_0} = \frac{\operatorname{erf}[(\xi/2(\alpha\theta)^{1/2}) + \lambda] - \operatorname{erf}\lambda}{\operatorname{erf}2\lambda - \operatorname{erf}\lambda} \tag{22}$$

Differentiating Eq. (22) with respect to ξ, substituting into Eqs. (9) and (11), and simplifying give

$$\frac{T_0 - T_m}{(\pi\alpha\theta)^{1/2}[\operatorname{erf}2\lambda - \operatorname{erf}\lambda]\exp\lambda^2} = \frac{h}{k}(T_g - T_0) \tag{23}$$

$$c(T_0 - T_m)/L(\pi)^{1/2} = \lambda[\operatorname{erf}2\lambda - \operatorname{erf}\lambda]\exp 4\lambda^2 \tag{24}$$

With these two equations, the unknowns T_0 and λ can be solved. Then the char depth is given by Eq. (19) and the ablation velocity by Eq. (20). The amount of material blown away is the area under the curve of ablation velocity vs time.

Virgin material layer

The governing equations are Eqs. (1, 7, and 8). As before, one of the solutions is

$$T_s = B_3 + B_4 \operatorname{erf}[x/2(\alpha_s\theta)^{1/2}] \tag{25}$$

The constants B_3 and B_4 can be determined from boundary conditions and Eqs. (7) and (8). The final result is

$$\frac{T_s - T_m}{T_i - T_m} = \frac{\operatorname{erf}[x/2(\alpha_s\theta)^{1/2}] - \operatorname{erf}[\lambda(\alpha_c/\alpha_s)^{1/2}]}{\operatorname{erf}[\delta_s/2(\alpha_s\theta)^{1/2}] - \operatorname{erf}[\lambda(\alpha_c/\alpha_s)^{1/2}]} \tag{26}$$

Discussion

If the ablative material is porous, the thermal diffusivity in the char-gas layer will be replaced by an effective thermal diffusivity defined by

$$\alpha_{\text{eff}} = k_{\text{eff}}/(\rho c)_{\text{eff}} \tag{27}$$

where

$$k_{\text{eff}} = k_g\epsilon + k_c(1 - \epsilon) \tag{28}$$

$$(\rho c)_{\text{eff}} = \epsilon\rho_g c_g + (1 - \epsilon)\rho_c c_c \tag{29}$$

Although these equations are developed on the basis of negligible transpiration and diffusion effects, they still can be applicable to take these effects into account by adjusting the heat-transfer coefficient h

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Closed-Form Lagrangian Multipliers for Coast Periods of Optimum Trajectories

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Nomenclature

r = radial distance from center of reference body
v = velocity
γ = complement of the flight-path angle
F = thrust
α = angle of attack
m = mass
g = acceleration due to gravity
λ_i = Lagrangian multipliers

1. Introduction

APPLICATION of the classical calculus of variations to trajectory optimization problems results, in general, in a system of simultaneous differential equations for the Lagrangian multipliers (essentially the direction cosines of the thrust vector) which, because of the nonexistence of or difficulty of obtaining an analytical solution, are integrated numerically to obtain values that determine the optimum

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